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THE BOSONIC STRING MEASURE AT TWO AND THREE LOOPS AND SYMPLECTIC TRANSFORMATIONS OF THE VOLUME FORM

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Abstract. Symplectic modular invariance of the bosonic string partition function has been verified at genus 2 and 3 using the period matrix coordinatization of moduli space. A calculation of the transformation of the holomorphic part of the differential volume element shows that an extra phase arises together with the factor associated with a specific modular weight; the phase is cancelled in the transformation of the entire volume element including the complex conjugate. An argument is given for modular invariance of the reggeon measure at genus twelve.

The formulas for the bosonic string partition function at two and three loops have been given as integrals of the type

$$Z_g \;=\; \int_{\mathcal{M}_g} \prod_{i=1}^{dim\ \mathcal{M}_g} dy_i \wedge d\bar{y}_i\ |F(y_i)|^2\ [det(Im\ \tau)]^{-13} \tag{1}$$

where the coordinates y_i may be represented by elements of the period matrix τ and $F(y_i)$ represents a section of a holomorphic bundle over moduli space [1]. The restriction to a single copy of moduli space is equivalent to integration over a fundamental domain, in the subspace of period matrices in the $\frac{1}{2}g(g+1)$-dimensional Siegel upper half space $\mathcal{H}_g$ $[det(Im\ \tau) > 0]$ corresponding to Riemann surfaces, defined with respect to the symplectic modular group Sp(2g; $\mathbb{Z}$)

$$\tau \;\rightarrow\; (A\tau + B)(C\tau + D)^{-1} \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g;\mathbb{Z}) \tag{2}$$

Then the function $F(y_i)$ can be expressed as a modular function of the period matrix [2][3][4], implying

$$\begin{aligned} Z_2 \;&=\; \int_{\mathcal{M}_2} \prod_{i=1}^{2} d\tau_{ij} \wedge d\bar{\tau}_{ij}\ |\chi_{10}(\tau)|^{-2}\ [det(Im\ \tau)]^{-13} \\ Z_3 \;&=\; \int_{\mathcal{M}_3} \prod_{i=1}^{3} d\tau_{ij} \wedge d\bar{\tau}_{ij}\ |\chi_{18}(\tau)|^{-1}\ [det(Im\ \tau)]^{-13} \end{aligned} \tag{3}$$

Given a symplectic transformation of the period matrix

$$\begin{aligned} Im\ \tau \;\rightarrow\; & \frac{1}{2i}[(A\tau + B)(C\tau + D)^{-1} - (A\bar{\tau} + B)(C\bar{\tau} + D)^{-1}] \\ =\; & \frac{1}{2i}(\tau C^T + D^T)^{-1}[(\tau A^T + B^T)(C\bar{\tau} + D)\ -\ (\tau C^T + D^T)(A\bar{\tau} + B)] \\ & \qquad \cdot (C\bar{\tau} + D)^{-1} \\ =\; & (\tau C^T + D^T)^{-1}\ (Im\ \tau)\ (C\bar{\tau} + D)^{-1} \end{aligned} \tag{4}$$

so that

$$[det(Im\ \tau)]^{-13} \;\rightarrow\; [det(Im\ \tau)]^{-13} \cdot |det(C\tau + D)|^{26} \tag{5}$$

Since

$$|\chi_{10}(\tau)|^{-2} \;\rightarrow\; |\chi_{10}(\tau)|^{-2}\ |det(C\tau + D)|^{-20} \tag{6}$$

modular invariance of the integrand in Z_2 depends on the transformation properties of the differential volume element. At genus two, the Jacobian of the coordinate transformation from $(\tau_{11},\tau_{12},\tau_{22})$ to $(\tau'_{11},\tau'_{12},\tau'_{22})$, $det\left(\frac{\partial(\tau'_{11},\tau'_{12},\tau'_{22})}{\partial(\tau_{11},\tau_{12},\tau_{22})}\right)$, where $\frac{\partial\tau'_{ij}}{\partial\tau_{kk}} = [A-(A\tau+B)(C\tau+D)^{-1}C]_{ik}(C\tau+D)^{-1}_{kj}$ and $\frac{\partial\tau'_{ij}}{\partial\tau_{12}} = [A-(A\tau+B)(C\tau+D)^{-1}]_{i1}(C\tau+D)^{-1}_{2j} + [A-(A\tau+B)(C\tau+D)^{-1}C]_{i2}(C\tau+D)^{-1}_{1j}$, equals

$$\begin{aligned} det&\begin{pmatrix} \frac{[A-(A\tau+B)(C\tau+D)^{-1}C]_{11}}{(C\tau+D)^{-1}_{12}} & \frac{[A-(A\tau+B)(C\tau+D)^{-1}C]_{12}}{(C\tau+D)^{-1}_{22}} \end{pmatrix} \\ &\quad det[A-(A\tau+B)(C\tau+D)^{-1}C]\ det(C\tau+D)^{-1} \\ &= det\begin{pmatrix} \frac{[A-(A\tau+B)(C\tau+D)^{-1}C]_{11}}{(C\tau+D)^{-1}_{12}} & \frac{[A-(A\tau+B)(C\tau+D)^{-1}C]_{12}}{(C\tau+D)^{-1}_{22}} \end{pmatrix} \\ &\quad det(C\tau+D)^{-2} \end{aligned} \tag{7}$$

as $det(A-(A\tau+B)(C\tau+D)^{-1}C) = det(AC^{-1}D-B)\ det(C\tau+D)^{-1}\ detC = det(C\tau+D)^{-1}$. Moreover, equality of the differentials $d\tau'_{12}$ and $d\tau'_{21}$ implies that

$$\begin{aligned} \frac{[A-(A\tau+B)(C\tau+D)^{-1}C]_{11}}{(C\tau+D)^{-1}_{11}} &= \frac{[A-(A\tau+B)(C\tau+D)^{-1}C]_{12}}{(C\tau+D)^{-1}_{21}} \\ &= \frac{[A-(A\tau+B)(C\tau+D)^{-1}C]_{21}}{(C\tau+D)^{-1}_{12}} \\ &= \frac{[A-(A\tau+B)(C\tau+D)^{-1}C]_{22}}{(C\tau+D)^{-1}_{22}} \end{aligned} \tag{8}$$

and the Jacobian becomes

$$\frac{[A-(A\tau+B)(C\tau+D)^{-1}C]_{11}}{(C\tau+D)^{-1}_{11}}\ det(C\tau+D)^{-3} \equiv \Phi_2(\tau)\ det(C\tau+D)^{-3} \tag{9}$$

The transformation of the holomorphic part of the differential volume element is then $\prod_{i\le j=1}^{2}\ d\tau_{ij}\ \to\ \prod_{i\le j=1}^{2} d\tau_{ij}\Phi_2(\tau)det(C\tau+D)^{-3}$. The factor $\Phi_2(\tau)$ represents a modification of earlier calculations of the transformations of the holomorphic part of the volume element [3]. It can be shown, however, that $\Phi_2(\tau)$ is a phase factor, because

$$A-(A\tau+B)(C\tau+D)^{-1}C\ =\ \Phi_2(\tau)[(C\tau+D)^{-1}]^T \tag{10}$$

from the relations (8) and the $\Phi_2(\tau)^2 = 1$ from the equality of the determinants. Consequently, the phase factor is cancelled by the complex conjugate in the transformation of $\prod_{i\le j=1}^{2}\ d\bar{\tau}_{ij}$ and

$$\prod_{i\le j=1}^{2}\ d\tau_{ij}\wedge d\bar{\tau}_{ij}\ \to\ \prod_{i\le j=1}^{2}\ d\tau_{ij}\wedge d\bar{\tau}_{ij}\ |det(C\tau+D)|^{-6} \tag{11}$$

Equations (5), (6) and (11) give the required modular invariance of the two-loop bosonic string partition function.

At three loops, the transformation from the coordinates $\{(\tau_{ij}),\ 1 \leq i \leq j \leq 3\}$ to $\{(\tau'_{ij}),\ 1 \leq i \leq j \leq 3\}$ gives rise to a Jacobian involving the sum of terms of the form

$$\prod_{\#[ij]=6} [A - (A\tau + B)(C\tau + D)^{-1}C]_{ij} \prod_{\#[kl]=6} (C\tau + D)^{-1}_{kl} \tag{12}$$

Since τ is a 3×3 matrix, one obtains

$$\prod_{i \leq j=1}^{3} d\tau_{ij} \ \rightarrow \ \prod_{i \leq j=1}^{3} d\tau_{ij}\ \Phi_3(\tau)\ det(C\tau + D)^{-4} \tag{13}$$

for the transformation of the holomorphic part of the differential volume element, where $\Phi_3(\tau)$ is a phase factor. Again, the phase factor represents a modification of the transformation rule which is cancelled by its complex conjugate in the action of the symplectic modular group on the total differential volume element

$$\prod_{i \leq j=1}^{3} d\tau_{ij} \wedge d\bar{\tau}_{ij} \ \rightarrow \ \prod_{i \leq j=1}^{3} d\tau_{ij} \wedge d\bar{\tau}_{ij}\ |det(C\tau + D)|^{-8} \tag{14}$$

Since

$$|\chi_{18}(\tau)|^{-1} \ \rightarrow \ |\chi_{18}(\tau)|^{-1}\ |det(C\tau + D)|^{-18} \tag{15}$$

modular invariance of the string integrand at three loops follows from equations (5), (14) and (15).

Thus, although a phase factor arises in the transformation of the holomorphic part of the differential volume element, it does not affect modular invariance of the measure. Moreover, equations (11) and (14) are special cases of a formula which holds for arbitrary genus.

$$\prod_{i \leq j=1}^{g} d\tau_{ij} \wedge d\bar{\tau}_{ij} \ \rightarrow \ \frac{\prod_{i \leq j=1}^{g}\ d\tau_{ij} \wedge d\bar{\tau}_{ij}}{|det(C\tau + D)|^{2(g+1)}} \tag{16}$$

follows from the invariance of the volume form

$$\frac{\bigwedge_{(i,j)=1}^{\frac{1}{2}g(g+1)}\ d\tau_{ij} \wedge d\bar{\tau}_{ij}}{[det(Im\ \tau)_{ij}]^{g+1}} \tag{17}$$

which may be deduced from the symplectic hermitian metric [5]

$$ds^2 \ = \ trace[(Im\ \tau_{ij})^{-1} d\tau_{ij} (Im\ \tau_{ij})^{-1} d\bar{\tau}_{ij}] \tag{18}$$

on the Siegel upper half space $\mathcal{H}_g$.

The volume form (17) can be obtained at low genus by pulling back the Hodge norm on E^{g+1}, where E is the Hodge line bundle over $\mathcal{M}_g$, to K, the determinant line bundle of the holomorphic cotangent bundle, using the isomorphism $K \simeq E^{g+1}$ valid for $\mathcal{M}_1,\ \mathcal{M}_2$, and $\mathcal{M}_3^0 = \mathcal{M}_3 - h_3$, with h_3 being the subvariety of genus-three hyperelliptic surfaces [5].

When $g > 3$, the dimension of the Siegel upper half space is greater than that of the moduli space $\mathcal{M}_g^0$, but a modular invariant measure on $\mathcal{M}_g^0$ can be defined by using the volume form resulting from the metric induced by the embedding of $\mathcal{M}_g^0$ into $\mathcal{H}_g$. It has already been noted that as the relation

$$\tau_{mn} \ = \ \frac{1}{2\pi i} \left[\ ln\ K_n\ \delta_{mn} \ + \ \sum_{\alpha} {}^{(m,n)}\ ln\ \left(\frac{\xi_{1m} - V_\alpha \xi_{1n}}{\xi_{1m} - V_\alpha \xi_{2n}} \frac{\xi_{2m} - V_\alpha \xi_{2n}}{\xi_{2m} - V_\alpha \xi_{1n}} \right) \right] \tag{19}$$

holds for any surface which can be uniformized by a Schottky group, it provides a coordinatization of the Schottky locus using the variables $K_n,\ \xi_{1n}$ and ξ_{2n} [6]. In particular, period matrices of the form (19), with restrictions placed on multipliers and fixed points to avoid overlapping of isometric circles, should provide solutions of the KP equation

$$\frac{3}{4} u_{yy} \ = \ \frac{\partial}{\partial x} \left(u_t - \frac{1}{4} u_{xxx} - \frac{3}{2} u u_x \right) \tag{20}$$

through

$$u(x,y,t) \ = \ 2{\partial_x}^2\ log\ \Theta(Ux + Vy + Wt + z_0 \mid \tau) \tag{21}$$

for some three-dimensional vectors $U,\ V,\ W \in \mathbb{C}^g$ and any $z_0 \in \mathbb{C}^g$ [7][8][9][10]. The Schottky locus can be described also by an infinite set of inequalities involving the period matrix defining the intersection of the fundamental region of the modular group $Sp(2g;\mathbb{Z})$ with the space of Riemann surfaces. At large genus, it has been shown that this set of inequalities can be reduced to a finite set in the degeneration limit $|K_n| \to 0$ leading to an exhaustion of the fundamental regions as g increases [11].

Using (19), the metric (18) can be re-expressed in terms of the Schottky group parameters. Since

$$
\begin{aligned}
d\tau_{nn} = \frac{1}{2\pi i}\Bigg[\frac{dK_n}{K_n} &+ \sum_\alpha{}^{(n,n)} \sum_m \frac{d}{dK_m}\, ln\, \left(\frac{\xi_{1n}-V_\alpha \xi_{1n}}{\xi_{1n}-V_\alpha \xi_{2n}}\frac{\xi_{2n}-V_\alpha \xi_{2n}}{\xi_{2n}-V_\alpha \xi_{1n}}\right)\, dK_m \\
&+ \sum_\alpha{}^{(n,n)} \sum_m \frac{d}{d\xi_{1m}}\, ln\, \left(\frac{\xi_{1n}-V_\alpha \xi_{1n}}{\xi_{1n}-V_\alpha \xi_{2n}}\frac{\xi_{2n}-V_\alpha \xi_{2n}}{\xi_{2n}-V_\alpha \xi_{1n}}\right)\, d\xi_{1m} \\
&+ \sum_\alpha{}^{(n,n)} \sum_m \frac{d}{d\xi_{2m}}\, ln\, \left(\frac{\xi_{1n}-V_\alpha \xi_{1n}}{\xi_{1n}-V_\alpha \xi_{2n}}\frac{\xi_{2n}-V_\alpha \xi_{2n}}{\xi_{2n}-V_\alpha \xi_{1n}}\right)\, d\xi_{2m}\Bigg]
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
\frac{d}{dK_m}\, ln\left(\frac{\xi_{1n}-T_m\xi_{1n}}{\xi_{1n}-T_m\xi_{2n}}\frac{\xi_{2m}-T_m\xi_{2n}}{\xi_{2n}-T_m\xi_{1n}}\right) &= \frac{\xi_{1n}-T_m\xi_{2n}}{\xi_{1n}-T_m\xi_{1n}}\frac{\xi_{2n}-T_m\xi_{1n}}{\xi_{2n}-T_m\xi_{2n}} \\
&\qquad \frac{d}{dK_m}\left[\frac{\xi_{1m}-T_m\xi_{1n}}{\xi_{1n}-T_m\xi_{2n}}\frac{\xi_{2n}-T_m\xi_{2n}}{\xi_{2n}-T_m\xi_{1n}}\right] \\
\frac{d}{dK_m}(\xi_{1n}-T_m\xi_{1n}) &= \frac{(\xi_{1n}-\xi_{1m})(\xi_{1n}-\xi_{2m})(\xi_{2m}-\xi_{1m})}{[(\xi_{1n}-\xi_{2m})-K_m(\xi_{1n}-\xi_{1m})]^2} \\
\frac{d}{dK_{m_j}} T_{m_1}...T_{m_l}(\xi_{1n}) &= T'_{m_1}(T_{m_2}...T_{m_l}(\xi_{1n}))T'_{m_2}(T_{m_3}...T_{m_l}(\xi_{1n}))\ ... \\
&\qquad T'_{m_{j-1}}(T_{m_j}...T_{m_l}(\xi_{1n})) \\
&\cdot\frac{(T_{m_{j+1}}...T_{m_l}(\xi_{1n})-\xi_{1m_j})(T_{m_{j+1}}...T_{m_l}(\xi_{1n})-\xi_{2m_j})(\xi_{1m_j}-\xi_{2m_j})}{[(T_{m_{j+1}}...T_{m_l}(\xi_{1n})-\xi_{2m_j})-K_{m_j}(T_{m_{j+1}}...T_{m_l}(\xi_{1n})-\xi_{1m_j})]^2} \\
when\ T_{m_{j+1}},\ ...,\ T_{m_l}\ &\neq\ T_{m_j}
\end{aligned}
\tag{23}
$$

the dependence of the following metric components on the multipliers can be verified

$$
g_{K_n,\bar{K}_n} \sim \frac{1}{|K_n|^2} \quad g_{K_n,\bar{\xi}_{1n}} \sim \frac{1}{K_n} \quad g_{K_n,\bar{\xi}_{2n}} \sim \frac{1}{K_n} \quad g_{\xi_{1n},\bar{K}_n} \sim \frac{1}{\bar{K}_n} \quad g_{\xi_{2n},\bar{K}_n} \sim \frac{1}{\bar{K}_n} \tag{24}
$$

Thus, to leading order, the multiplier part of the integration measure at genus g constructed from such a metric will be

$$
\frac{d^2K_1}{|K_1|^2}\ \frac{d^2K_2}{|K_2|^2}...\ \frac{d^2K_g}{|K_g|^2}\ \frac{1}{[det(Im\ \tau)]^{g+1}} \tag{25}
$$

This resembles part of the superstring measure [12] and finiteness of superstring amplitudes [13][14] is consistent with finiteness of the symplectic volume of $\mathcal{H}_g/Sp(2g;\mathbb{Z})$ [5][15].

It is of particular interest to compare this measure at genus twelve with the Polyakov measure resulting from the Mumford isomorphism of bundles over moduli space

$K \simeq E^{13}$ [16]. Using the reggeon calculus formalism [17], a moduli space measure has been constructed

$$\prod_{n=1}^{g} \frac{d^2K_n}{|K_n|^4}|1-K_n|^4 \; \frac{1}{Vol(SL(2,\mathbb{C}))} \prod_{m=1}^{g} \frac{d^2\xi_{1m}d^2\xi_{2m}}{|\xi_{1m}-\xi_{2m}|^4} \; [det(Im\;\tau)]^{-13}$$
$$\prod_{\alpha}{}' \prod_{p=1}^{\infty} |1-K_\alpha^p|^{-48} \prod_{\alpha}{}' |1-K_\alpha|^{-4} \tag{26}$$

which is consistent with the singularity and harmonic properties characteristic of the Polyakov measure [1]. Equivalence of (26) with the Polyakov measure at genus two and three, expressed in terms of theta functions in equation (3), and consequently modular invariance of the two- and three-loop reggeon measure, has been established [4]. However, as a direct proof of equivalence at higher genus has not been feasible, because of the absence of a similar representation in terms of period matrices, modular invariance of the measure (26) at higher genus remains to be shown.

Nevertheless, at genus twelve, the Polyakov measure differs from the measure induced by the symplectic metric (18), $d\mu_{12}^{symp}$, containing the terms in equation (25), by a factor $|F_1|^{-2}$, which is singular at the boundary of moduli space. Since the reggeon measure (26) has the same singularity properties as the Polyakov measure in the limit of degenerate Riemann surfaces, it will differ from $d\mu_{12}^{symp}$ by the same factor $|F_1(K_n,\xi_{1n},\xi_{2n})|^{-2}$ up to the square of the absolute value of an analytic function $|F_2(K_n,\xi_{1n},\xi_{2n})|^2$, defined on a subset of moduli space $\mathcal{M}_{12}^0$ [5].

Now consider a modular transformation of the measure (26). The singularity properties of the measure and therefore $|F_1|^{-2}$ will be unchanged. The transformed reggeon measure is given by the product of $d\mu_{12}^{symp}$ and $|F_1|^{-2}|\tilde{F}_2|^2$, as $d\mu_{12}^{symp}$ is modular invariant. Since the ratio $\left|\frac{\tilde{F}_2(K_n,\xi_{1n},\xi_{2n})}{F_2(K_n,\xi_{1n},\xi_{2n})}\right|^2$ can be regarded as the square of the absolute value of an analytic function on all of moduli space, spanned by the Schottky group coordinates, it may be set equal to one after a suitable normalization of the measure. This argument suggests, therefore, that the expected modular invariance of the moduli space measure derived from the reggeon formalism is valid at genus twelve.

In conclusion, at genus two and three, there are two isomorphisms and $K \simeq E^{13}$ and $K \simeq E^{g+1}$ giving rise to two different modular invariant measures (3) and (17) respectively, involving the period matrix coordinatization of moduli space. Modular invariance of the two- and three-loop measures (3) follows because the factor arising in the transformation $\prod_{i \le j} \ d\tau_{ij}$ is only an extra phase. Even in the absence of an isomorphism $K \simeq E^{g+1}$ at higher genus, a modular-invariant measure (17), obtained from the symplectic metric in the Siegel upper half space $\mathcal{H}_g$, can be re-expressed in terms of Schottky group parameters and compared directly with the reggeon measure.

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